Year 13 Mathematics

IAS 3.1

Conic Sections

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Contents

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NCEA 3 Internal Achievement Standard 3.1 – Conic Sections

This achievement standard involves applying the geometry of conic sections in solving problems.

Achievement	Achievement with Merit	Achievement with Excellence
• Apply the geometry of conic sections in solving problems.	• Apply the geometry of conic sections, using relational thinking, in solving problems.	• Apply the geometry of conic sections using extended abstract thinking, in solving problems.

◆ This achievement standard is derived from Level 8 of The New Zealand Curriculum and is related to the achievement objectives
  ❖ apply the geometry of conic sections
  in the Mathematics strand of the Mathematics and Statistics Learning Area.

◆ Apply the geometry of conic sections in solving problems involves:
  ❖ selecting and using methods
  ❖ demonstrating knowledge of concepts and terms
  ❖ communicating using appropriate representations.

◆ Relational thinking involves one or more of:
  ❖ selecting and carrying out a logical sequence of steps
  ❖ connecting different concepts or representations
  ❖ demonstrating understanding of concepts
  ❖ forming and using a model;
  and relating findings to a context, or communicating thinking using appropriate mathematical statements.

◆ Extended abstract thinking involves one or more of:
  ❖ devising a strategy to investigate or solve a problem
  ❖ identifying relevant concepts in context
  ❖ developing a chain of logical reasoning, or proof
  ❖ forming a generalisation;
  and using correct mathematical statements, or communicating mathematical insight.

◆ Problems are situations that provide opportunities to apply knowledge or understanding of mathematical concepts and methods. Situations will be set in real-life or mathematical contexts.

◆ Methods include a selection from those related to:
  ❖ graphs and equations of the circle, ellipse, parabola and hyperbola
  ❖ Cartesian and parametric forms
  ❖ properties of conic sections
  ❖ tangents and normals.

The Circle

The Equation of a Circle

A circle is the path or locus of points which are an equal distance from a given point, the centre.

When the centre of the circle is the origin then the equation of the circle radius r is given by

$$x^2 + y^2 = r^2$$

When the centre is (a, b) then the equation becomes

$$(x - a)^2 + (y - b)^2 = r^2$$

Completing the Square

If the equation has been expanded

e.g. $x^2 + y^2 - 6x + 4y + 4 = 0$

then we need to factorise it (complete the square) in order to find the values of a, b and r.

We do this by first grouping all the variables on the left while leaving the constants on the right.

$$x^2 + y^2 - 6x + 4y + 4 = 0$$

$$x^2 - 6x + y^2 + 4y = {}^{-}4$$

We now make each of the two variables into a perfect square by using the pattern

$$(x \pm a)^2 = x^2 \pm 2ax + a^2$$

We compare

$$x^2 - 6x + ? = x^2 \pm 2ax + a^2$$

to identify the value for a (in this case $^{-}3$ see example next page) and replace ? with a^2 (in this case $({}^{-}3)^2 = 9$) to both sides of the equation to complete the pattern.

We compare

$$y^2 + 4y + ? = y^2 \pm 2ay + a^2$$

to identify the value for a (in this case 2 see example next page) and replace ? with a^2 (in this case $(2)^2 = 4$) to both sides of the equation to complete the pattern.

Therefore

$$x^2 - 6x + 9 + y^2 + 4y + 4 = {}^{-}4 + 9 + 4$$

$$(x - 3)^2 + (y + 2)^2 = 9$$

$$(x - 3)^2 + (y + 2)^2 = 3^2$$

We can now read off the centre and radius of the circle, i.e. centre $(3, {}^{-}2)$, radius 3.

Circles, Ellipses, Parabolas etc. are called conic sections as they are formed when we cut through a cone.

A circle is formed when we cut straight across the the cone.

Both the Casio 9750GII and the TI-84 Plus can graph conics. Detailed below are the instructions related to circles, but you can also use your graphics calculator, in a similar way for the other conics studied in this chapter, i.e. ellipse, parabola and hyperbola.

On the Casio 9750GII from the main MENU select CONICS $\boxed{\text{MENU}}$ $\boxed{7}$. You are presented with a number of options.

Scroll down to

$(X - H)^2 + (Y - K)^2 = R^2$ or

$AX^2 + AY^2 + BX + CY + D = 0$

An alternative to completing the square is to graph the equation given in expanded form

e.g. $x^2 + y^2 - 6x + 4y + 4 = 0$

You select

$AX^2 + AY^2 + BX + CY + D = 0$ form

and enter A = 1, B = $^{-}6$, C = 4 and D = 4.

$\boxed{1}$ $\boxed{\text{EXE}}$ $\boxed{(-)}$ $\boxed{6}$ $\boxed{\text{EXE}}$ $\boxed{4}$ $\boxed{\text{EXE}}$ $\boxed{4}$

$\boxed{\text{EXE}}$ $\overset{\text{DRAW}}{\boxed{\text{F6}}}$ $\boxed{\text{SHIFT}}$ $\overset{\text{G-Solv}}{\boxed{\text{F5}}}$ $\overset{\text{CNTR}}{\boxed{\text{F1}}}$ OR $\overset{\text{RADS}}{\boxed{\text{F2}}}$

Select Graph Solve to find the radius (3) and centre $(3, {}^{-}2)$.

You are then able to use these values to write the equation in the form

$$(x - 3)^2 + (y + 2)^2 = 3^2$$

On your TI-84 Plus you access the conics menu via the application button and scroll down until you get to the conics sub-menu.

$\boxed{\text{APPS}}$ $\boxed{\blacktriangledown}$ $\boxed{\text{ENTER}}$ $\overset{\text{CIRCLE}}{\boxed{\text{ENTER}}}$

After selecting circle choose the form of the equation you require.

1: $(X - H)^2 + (Y - K)^2 = R^2$

2: $AX^2 + AY^2 + BX + CY + D = 0$

In this case $\boxed{2}$. Enter

$\boxed{1}$ $\boxed{\text{ENTER}}$ $\boxed{(-)}$ $\boxed{6}$ $\boxed{\text{ENTER}}$ $\boxed{4}$ $\boxed{\text{ENTER}}$ $\boxed{4}$

$\boxed{\text{ENTER}}$ $\boxed{\text{GRAPH}}$

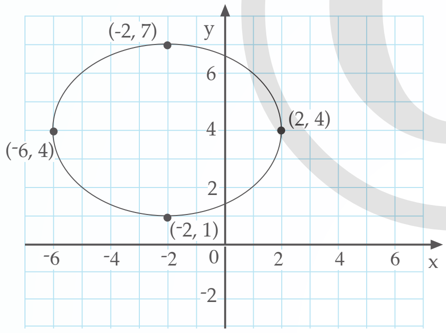
Example

Find the coordinates of the centre, the length of the major and minor axes and the coordinates of the foci. Sketch the ellipse.

$$\frac{(x+2)^2}{16}+\frac{(y-4)^2}{9}=1$$

The centre of this ellipse is $(^{-}2, 4)$ and $a = 4$ and $b = 3$
Length of major axis is $(2 \times a) = 8$
Length of minor axis is $(2 \times b) = 6$

To find the foci we use
$c^2 = a^2 - b^2$,
so $c = 2.6$
Foci are $(^{-}2 - 2.6, 4)$ and $(^{-}2 + 2.6, 4)$
i.e. $(^{-}4.6, 4)$ and $(0.6, 4)$.



Example

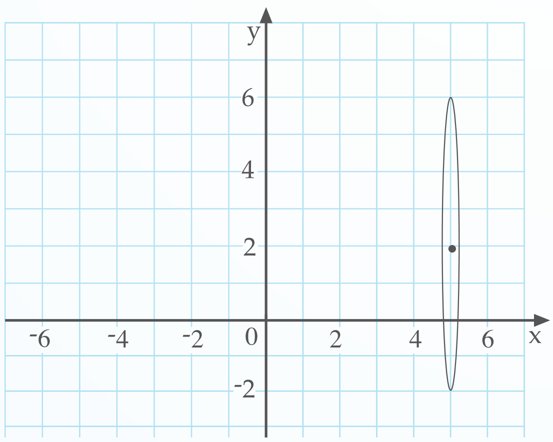
Write down the Cartesian equation of the ellipse with parametric equations

$$x = 5 + \frac{1}{4}\cos t \text{ and } y = 2 - 4\sin t$$

Centre is $(5, 2)$ with $a = \frac{1}{4}$ and $b = {}^{-}4$

$$\frac{(x-5)^2}{\left(\frac{1}{4}\right)^2}+\frac{(y-2)^2}{(^{-}4)^2}=1$$

$$16(x-5)^2+\frac{(y-2)^2}{16}=1$$



Example

Show that

$$25x^2 + 4y^2 - 200x + 16y + 316 = 0$$

is the equation of an ellipse and find its centre and the length of each axis.

$$25x^2 + 4y^2 - 200x + 16y + 316 = 0$$

Collect like terms together and factorise each term

$$25(x^2 - 8x) + 4(y^2 + 4y) + 316 = 0$$

Complete the square for each variable. The term being subtracted is to balance what has been added.

$$25(x^2 - 8x + 16 - 16) + 4(y^2 + 4y + 4 - 4) + 316 = 0$$

$$25((x-4)^2 - 16) + 4((y+2)^2 - 4) + 316 = 0$$

$$25(x-4)^2 - 400 + 4(y+2)^2 - 16 + 316 = 0$$

$$25(x-4)^2 + 4(y+2)^2 = 100$$

Divide through by 100 to get it into the form of an ellipse.

$$\frac{25(x-4)^2}{100}+\frac{4(y+2)^2}{100}=1$$

$$\frac{(x-4)^2}{4}+\frac{(y+2)^2}{25}=1$$

Ellipse, centre $(4, {}^{-}2)$ with major axis $(2 \times 5) = 10$ and minor axis $(2 \times 2) = 4$.

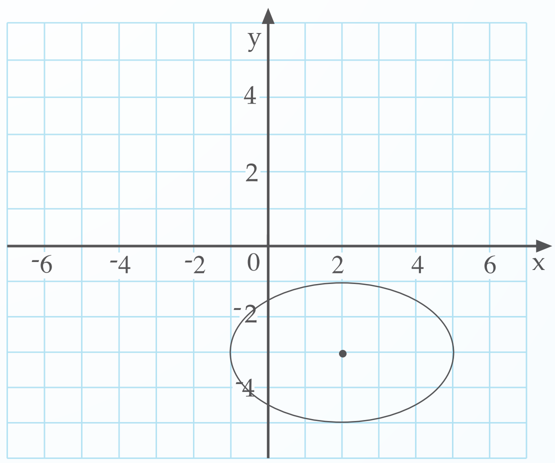
Example

Write down a pair of parametric equations for the

ellipse $$\frac{(x-2)^2}{9}+\frac{(y+3)^2}{4}=1$$

Centre is $(2, {}^{-}3)$ with $a = 3$ and $b = 2$

$x = 2 + 3\cos t$ and $y = {}^{-}3 + 2\sin t$



Example

Give the equation of the parabola with vertex (3, 2), axis of symmetry y = 2 and a point (5, 6) which is on the curve. Locate its focus.

Using

$$(y - y_1)^2 = 4a(x - x_1)$$

Vertex of the parabola is (3, 2) therefore

$$(y - 2)^2 = 4a(x - 3)$$

Substituting (5, 6) to find 'a'

$$(6 - 2)^2 = 4a(5 - 3)$$

$$16 = 8a$$

$$a = 2$$

So equation of the parabola is

$$(y - 2)^2 = 8(x - 3)$$

Using a = 2 to find the focus

$$\text{Focus} = (3 + a, 2)$$

$$= (5, 2)$$

Example

Show that

$$y^2 - 4x + 2y + 5 = 0$$

is the equation of a parabola and sketch the graph. Locate its vertex, focus and the directrix.

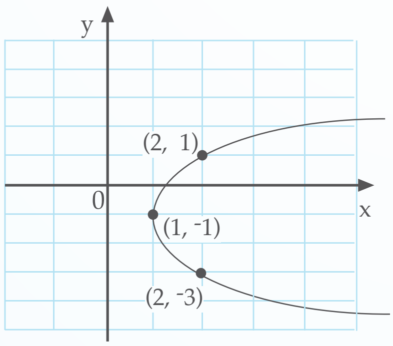
Rearranging and completing the square.

$$\begin{aligned} y^2 - 4x + 2y + 5 &= 0 \\ y^2 + 2y &= 4x - 5 \\ (y + 1)^2 - 1 &= 4x - 5 \\ (y + 1)^2 &= 4x - 4 \\ (y + 1)^2 &= 4(x - 1) \end{aligned}$$

Vertex $(1, {}^-1)$ and focus a = 1

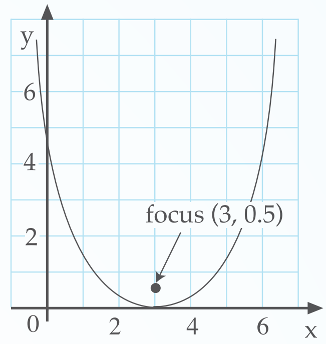
focus = $(2, {}^-1)$

Equation of the directrix is x = 0



Example

A parabola has a vertex at (3, 0) and focus at (3, 0.5), find its equation and y intercept.



As the parabola is arranged vertically its equation must be in the form

$$\begin{aligned} (x - x_1)^2 &= 4a(y - y_1) \\ \text{vertex} &= (3, 0) \\ a &= 0.5 \\ (x - 3)^2 &= 4 \times 0.5(y - 0) \\ (x - 3)^2 &= 2y \end{aligned}$$

y intercept (x = 0)

$$\begin{aligned} (0 - 3)^2 &= 2y \\ y &= 4.5 \end{aligned}$$

Example

A parabola has parametric equations

$$y = 1 + 4t \text{ and } x = t^2 - 2$$

Find the Cartesian equation of the parabola and its y intercepts. Sketch the graph.

Making t the subject.

$$t = \frac{y-1}{4}$$

Substituting $\quad x = \left(\dfrac{y-1}{4}\right)^2 - 2$

$$16(x + 2) = (y - 1)^2$$

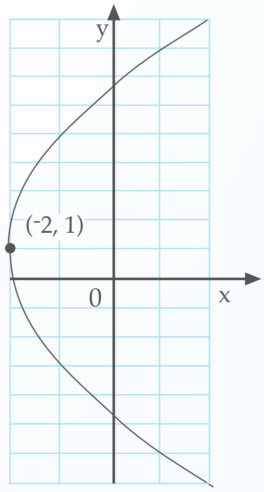
$$(y - 1)^2 = 16(x + 2)$$

which is the form of the parabola $y^2 = 4ax$ with vertex translated to $({}^-2, 1)$.

y intercepts

x = 0 $\quad y = 6.66$

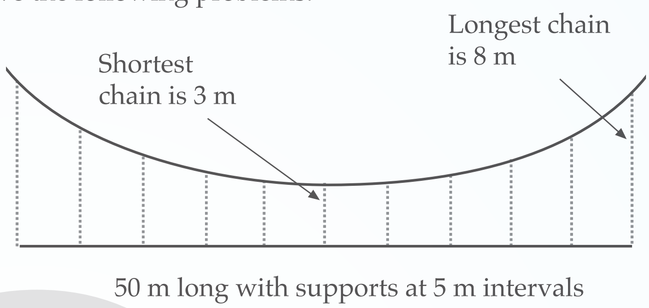
and $\quad y = {}^-4.66$



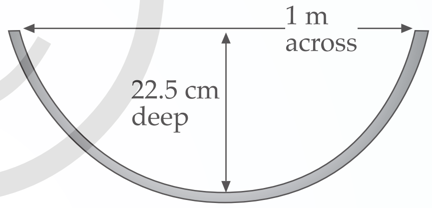
Merit – Use your knowledge of conics to solve the following problems.

81. The wire supporting a suspension bridge is best modelled by a parabola. It is connected to the bridge by chain lengths. The first chain is 8 m and the shortest length is 3 m. If they are placed at 5 m intervals and the bridge is 50 m long find the length of chain used.

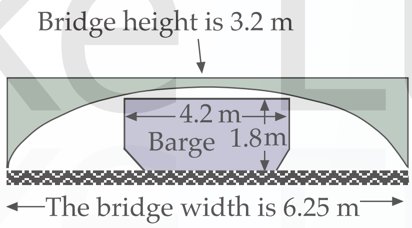
Hint: Draw some axes on the diagram that will minimise your work.



82. A student constructing the parabolic reflector of a satellite dish is working to the following measurements. Find the position of the focus.



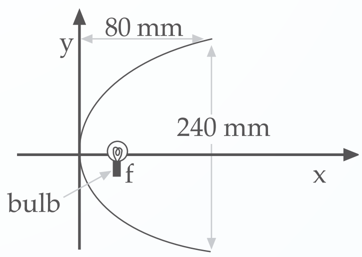
83. The span on the underside of a bridge can be modelled by a parabola with a maximum height of 3.2 m and width of 6.25 m. Investigate if a barge with dimensions above the waterline of 4.2 m across and 1.8 m high (includes a gap for clearance) can fit under the bridge.



84. A car headlight is in the shape of a parabolic dish. The bulb is placed at the focus (f). The headlight is 240 mm across the front and 80 mm at its greatest depth.

a) Find the distance from the vertex of the parabolic dish to the bulb.

b) Find the pair of parametric equations for the parabola.



The Equations of Tangents and Normals

Tangents and Normals using Implicit Differentiation

A **tangent** is a line that touches a curve at a point without crossing it.

A **normal** is a line that is perpendicular to a tangent at a point on a curve.

To find the equation of a tangent or normal it is usually necessary to differentiate the function concerned first, to find the slope (gradient) at the designated point.

If the equation includes a y^2 component, this will require differentiating implicitly.

Earlier we used the formula

$$y - y_1 = m(x - x_1)$$

to find the equation of the desired straight line with tangent m.

We also used the relationship between the gradients of perpendicular lines

$$m_1 m_2 = {}^{-}1$$

OR $$m_1 = \frac{^{-}1}{m_2}$$

which will be needed here.

Implicit Differentiation

Implicit differentiation is explained in the NuLake EAS 3.6 Booklet or our EAS Calculus Workbook. You will need to understand this if you want to reach Merit level, although it is possible to differentiate using the parametric form (see examples below and later in this section).

Example

Find the equation of the tangent to the circle $x^2 + y^2 = 29$ at the point P (5, 2).

Differentiating the circle implicitly

$$x^2 + y^2 = 29$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = {}^{-}2x$$

$$\frac{dy}{dx} = \frac{^{-}x}{y}$$

At the point (5, 2) the gradient of the tangent is $\frac{^{-}5}{2}$.

Tangent is $$y - 2 = \frac{^{-}5}{2}(x - 5)$$

$$2y - 4 = {}^{-}5x + 25$$

$$2y + 5x - 29 = 0$$

Example

Find the equation of the normal to the parabola $y^2 = 8(x - 2)$ at (4, $^{-}4$) and find where this normal intersects the curve again.

Differentiating implicitly

$$y^2 = 8(x - 2)$$

$$y^2 = 8x - 16$$

$$2y\frac{dy}{dx} = 8$$

$$\frac{dy}{dx} = \frac{4}{y}$$

At the point (4, $^{-}4$) the gradient of the tangent is $^{-}1$.

Gradient of the normal is 1.

Normal is $$y + 4 = 1(x - 4)$$

$$y + 4 = x - 4$$

$$y = x - 8$$

Substitute for y $$(x - 8)^2 = 8(x - 2)$$

$$x^2 - 16x + 64 = 8x - 16$$

gives $$x^2 - 24x + 80 = 0$$

$$(x - 4)(x - 20) = 0$$

$$x = 4, 20$$

The point of intersection is (20, 12)

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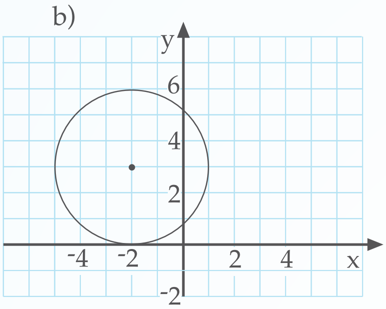
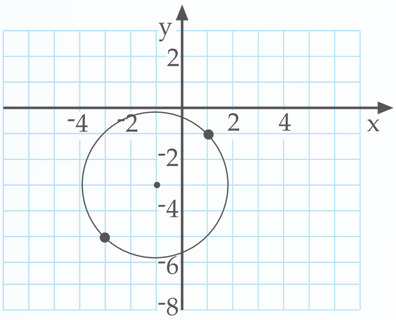
35. a) $x = {}^{-}2 + 3\cos t$
$y = 3 + 3\sin t$

b)

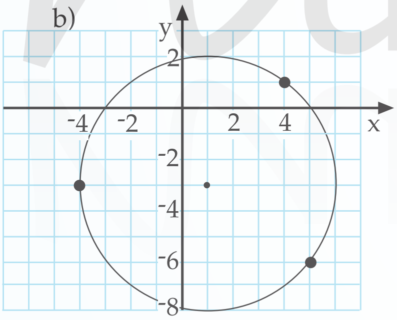


36. $x = {}^{-}1 + 2\sqrt{2}\ \cos t$
$y = {}^{-}3 + 2\sqrt{2}\ \sin t$



37. a) $x = 1 + 5\cos t$
$y = {}^{-}3 + 5\sin t$

b)



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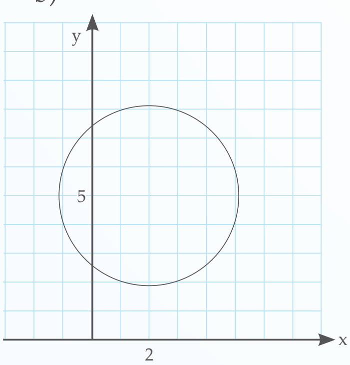
38. a) $(x - 2)^2 + (y - 5)^2 = 10$

b)



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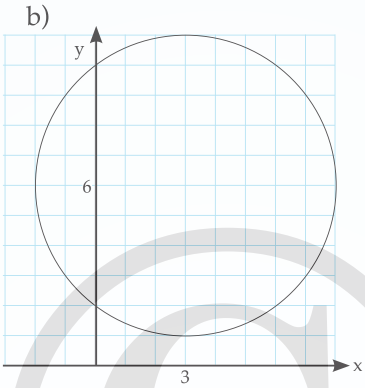
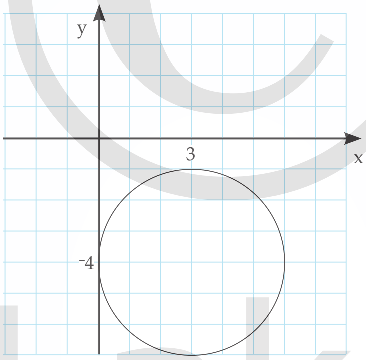
39. a) $x = 3 + 5\cos t$
$y = 6 + 5\sin t$

b)

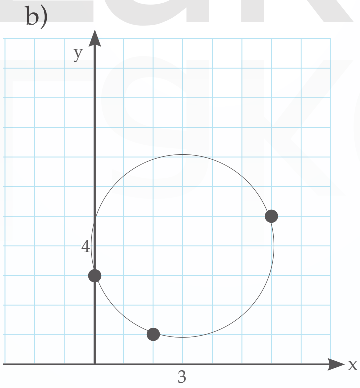


40. $x = 3 + 3\cos t$
$y = {}^{-}4 + 3\sin t$



41. a) $x = 3 + \sqrt{10}\ \cos t$
$y = 4 + \sqrt{10}\ \sin t$

b)



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42. Centre (0, 0)
Major axis = 12
Minor axis = 10
Foci $({}^{-}3.32, 0)$ and (3.32, 0)
End points (±6, 0) and (0, ±5)

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43. Centre $({}^{-}2, 3)$
Major axis = 10
Minor axis = 6
Foci $({}^{-}6, 3)$ and (2, 3)
End points $({}^{-}7, 3)$ to (3, 3)
End points $({}^{-}2, 0)$ to $({}^{-}2, 6)$

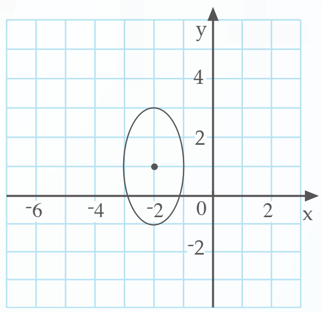
44. Centre $(2, {}^{-}1)$
Major axis = 6
Minor axis = 2
Foci $({}^{-}0.83, {}^{-}1)$ and $(4.83, {}^{-}1)$
End points $({}^{-}1, {}^{-}1)$ to $(5, {}^{-}1)$
End points (2, 0) to $(2, {}^{-}2)$

45. Centre (0, 0)
Major axis = $2\sqrt{5}$
Minor axis = 4
Foci $({}^{-}1, 0)$ and (1, 0)
End points $(\pm\sqrt{5}, 0)$
End points (0, ±2)

46. Centre $({}^{-}2, 5)$
Major axis = 2
Minor axis = 1
Foci $({}^{-}2.87, 5)$ and $({}^{-}1.13, 5)$
End points $({}^{-}3, 5)$ to $({}^{-}1, 5)$
End points $({}^{-}2, 4.5)$ to $({}^{-}2, 5.5)$

47. Centre (4, 0)
Major axis = 8
Minor axis = 6
Foci (1.35, 0) and (6.65, 0)
End points (0, 0) to (8, 0)
End points $(4, {}^{-}3)$ to (4, 3)

48. $\dfrac{(x+2)^2}{1} + \dfrac{(y-1)^2}{4} = 1$
Centre $({}^{-}2, 1)$
Foci $({}^{-}2, {}^{-}0.73)$ and $({}^{-}2, 2.73)$



49. $\dfrac{(x+1.5)^2}{3.5^2} + \dfrac{(y-1)^2}{1} = 1$

$x = {}^{-}1.5 + 3.5\cos t$
$y = 1 + \sin t$

Foci $({}^{-}4.85, 1)$ and (1.85, 1)